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Comparison between the Jacobi and Gauss-Seidel methods for solving partial differential equations

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Abstract:

This study aimed to search for a solution to linear partial differential equations, which represent many physical and engineering phenomena, and which only appear in the form of mathematical systems that describe the nature of these phenomena. The focus here was on the Laplace equation in the second dimension, as a model for describing these phenomena using the finite difference method. To approximate the solution of these equations, the equation is converted to another form, so that a linear system is obtained, which can be solved using one of the iterative methods. Among the methods used in this paper are the Jacobi and Gauss-Seidel methods, and it was found from what the numerical results showed that the Gauss method - Seidel is one of the best iterative methods to obtain a fast and approximate solution to the exact solution, which gives the lowest possible error.

Keywords: linear partial differential equations, the Jacobi method, the Gauss-Seidel method

1. Introduction:

Differential equations in general and partial equations in particular are characterized by their great importance in their applications that include the fields of science and knowledge, as they are a link between mathematics, physics, and other sciences, as differential equations generate physically absolutely from their laws such as Newton's laws in mechanics, Kirchhoff's law in electricity, and Maxwell's laws in electromagnetism. And others, and as a result of their connection to issues related to various fields of life, many scientific studies have appeared that investigate the solutions of these equations, either numerically or analytically. We find that the solutions of these equations must meet certain boundary conditions, within the scope of the scope that governs them, such as boundary value problems. Which are divided into two types, the first are free boundary problems, which are related to solving incomplete partial differential equations, which are represented by stable state problems, and the second are moving boundary problems, which are related to the equivalent and hyperbolic partial equations, which are represented by the equations of propagation, vibration motion, and waves [1, 3, 7].

Partial differential equations are equations that include more than one independent variable, and the most famous of them is the linear partial differential equation of the second order and its general form:

$$A\varphi_{xx} + B\varphi_{xy} + C\varphi_{yy} + D\varphi_x + E\varphi_y + F\varphi = G(x,y) \tag{1}$$

Where A, B, C, D constant.

$$\varphi_{xx} = \frac{\partial^2 \varphi}{\partial x^2}, \qquad \varphi_{yy} = \frac{\partial^2 \varphi}{\partial y^2}, \qquad \varphi_{xy} = \frac{\partial^2 \varphi}{\partial y \partial x}$$

Every partial differential equation such as (1) represents one of the following types, which are defined as follows:

Equivalent equation: These are equations that describe heat flow and diffusion processes and are verified

$$B^2 - 4AC > 0$$

An incomplete equation that describes steady-state phenomena

$$B^2 - 4AC < 0$$

Hyperbolic equation, which are equations that describe vibration movements and wave movements and are verified

$$B^2 - 4AC = 0$$

Linear partial differential equations are used in many applied problems, including the wave equation, the heat flow equation, the Laplace equation, the Poisson equation, and others. There are very many scientifically applied methods, most of which are methods that follow them to transform partial differential equations into ordinary differential equations, whether analytical or numerical. Analytical methods include: separation of variables, integral transformations, transposition of variables, tremor method, incentive method, and other methods. As for numerical methods, the partial differential equation is transformed into a large set of simpler equations, applied to infinitesimal parts or points, taking into account the conditions. Primary or borderline, where parts or points are connected, and these methods are many and many, the most important of which are the finite difference method, the method of finite elements and finite volumes. We will suffice with using the finite difference method [6, 8, 10].

2. Finite Difference Method:

One of the iterative methods commonly used in solving partial differential equations for boundary value problems is to replace the terms of the partial derivatives with their corresponding finite difference approximations. The solution requires dividing the domain of the problem into a grid consisting of a set of rectangles or squares. The grid may be regular or surrounded by a curve. The smaller the divisions, the more accurate the results [2, 5, 11].

If we consider the rectangular region R in the plane xy , we divide this region into a grid of rectangles such that $\Delta x = h, \Delta y = k,$ the points of intersection of the lines are called grid points[8].

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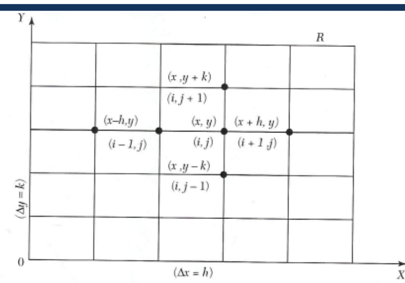

Fig.1 shows the network numbering method [6].

So we have finite difference approximations of the partial derivatives in one direction [7,8]:

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \qquad and \qquad u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

by approximating the finite differences, we obtain:

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}\right] \qquad (2)$$

3. Iterative Method for solving Linear system.

3.1 Jacobi method:

The Jacobi Method is one of the easiest iterative method for solving linear equations, and is given by the sequence [4, 9]:

$$u_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{i=1}^{n} -a_{ii}\, u_j^{(k-1)} + b_i\right], \qquad i \neq j, a_{ii} \neq 0, i = 1,2,3,..,n \qquad (3)$$

Where $k \in N^*$ and n is the number of the unknown variables[4].

3.2 Gauss-Seidel method:

It is used to solve systems of linear equations in which most of the coefficients are zero, and is given by the sequence [4, 9]:

$$u_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{j=1}^{i-1} a_{ij}\, u_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k-1)} + b_i\right], \quad i \neq j, a_{ii} \neq 0, i = 1,2.,n \qquad (4)$$

4. Laplace Equation and Problem

Defined Laplace equation by two diminutions at the following:

$$u_{xx} + u_{yy} = 0 \qquad (5)$$

Consider a rectangular region R. Whose

$$R = \{(x, y) : a = 0 < x < b = 1, c = 0 < y < d = 1\}$$

And boundary conditions given on the boundaries

$$u(x,1) = x, \quad u(1,y) = y, \quad u(0,y) = u(x,0) = 0$$

We want to approximate the solution u by using Finite difference

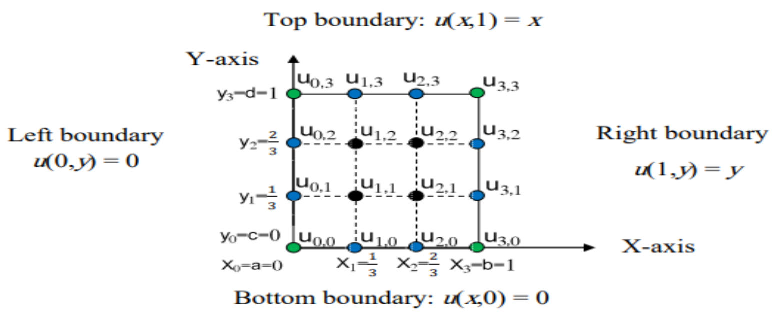

Fig.2 Explains the boundary conditions [6].

From the figure we conclude that:

$$n = m = 3, \qquad h = \frac{1}{3}, \qquad k = \frac{1}{3}$$

We use the finite difference equation (2) to approximate the internal grid points, and rearranging we get the following equations:

$$4u_1 - u_2 - u_3 = \frac{1}{3}$$

$$-u_1 + 4u_2 - u_4 = \frac{4}{3}$$

$$-u_1 + 4u_3 - u_4 = 0$$

$$-u_2 - u_3 + 4u_4 = \frac{1}{3}$$

We put the previous equations in the form of a following:

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 4/3 \\ 0 \\ 1/3 \end{bmatrix}$$

If we apply Gaussian elimination to solve this linear system, we get the exact solution

$$u = (0.222222, 0.444444, 0.111111, 0.222222)^T$$

Now: we will solve these linear equations by iterative method like Jacobi method, Gauss-Seidel

From the equation (3) (**Jacobi method):**

$$u_1^{(k)} = \frac{1}{4}u_2^{(k-1)} + \frac{1}{4}u_3^{(k-1)} + \frac{1}{12}$$

$$u_2^{(k)} = \frac{1}{4}u_1^{(k-1)} + \frac{1}{4}u_4^{(k-1)} + \frac{1}{3}$$

$$u_3^{(k)} = \frac{1}{4}u_1^{(k-1)} + \frac{1}{4}u_4^{(k-1)}$$

$$u_4^{(k)} = \frac{1}{4}u_2^{(k-1)} + \frac{1}{4}u_3^{(k-1)} + \frac{1}{12}$$

Consider the initial solution is $\left(u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, u_4^{(0)}\right)^T = (0,0,0,0)^T$

For $(k = 1)$ We get the first iteration:

$$u_1^{(1)} = \frac{1}{12}, \quad u_2^{(1)} = \frac{1}{3}, \quad u_3^{(1)} = 0, \quad u_4^{(1)} = \frac{1}{12}$$

The following approximate solution is found by Matlab program for the sixteen iterations. (see Appendix A).

$u = (0.222219, 0.444440, 0.111107, 0.222219)^T$

Now: From the equation (4) **Gauss-Seidel method:**

$$u_1^{(k)} = \frac{1}{4}u_2^{(k-1)} + \frac{1}{4}u_3^{(k-1)} + \frac{1}{12}$$

$$u_2^{(k)} = \frac{1}{4}u_1^{(k)} + \frac{1}{4}u_4^{(k-1)} + \frac{1}{3}$$

$$u_3^{(k)} = \frac{1}{4}u_1^{(k)} + \frac{1}{4}u_4^{(k-1)}$$

$$u_4^{(k)} = \frac{1}{4}u_2^{(k)} + \frac{1}{4}u_3^{(k)} + \frac{1}{12}$$

For $(k = 1)$ We get the first iteration:

$$u_1^{(1)} = \frac{1}{12}, \quad u_2^{(1)} = \frac{17}{48}, \quad u_3^{(1)} = \frac{1}{48}, \quad u_4^{(1)} = \frac{33}{192}$$

The following approximate solution is found by Matlab program for the first nine iterations (see Appendix B)

5. Comparison between the iterative method:

The generated linear system in the example above, which must be solved by some iterative methods, namely Jacobi and Gauss-Seidel. The following table shows the numerical results of these iterative methods, as each of them obtains the approximate solution with a different number of iterations, leading to fewer errors and accurate solutions.

Table 1 shows the iterative method and the number iterations

The exact solution is $u = (0.222222, 0.444444, 0.111111, 0.222222)^T$					
Method					Number Of iterations
Jacobi solution	0.222219	0.444440	0.111107	0.222219	16
Gauss-Seidel solution	0.222219	0.444443	0.111110	0.222222	9

6. Conclusion:

This research deals with solving a type of partial differential equation and applying it to physical problems that involve two independent variables, we used the finite difference method to solve boundary value problems that include the Laplace equation, where a linear system of algebraic equations is obtained, which is solved repeatedly through different iterative schemes, including: Jacobi and Gauss-Seidel. We noticed that the finite difference method is an effective and simple method. When the field is very regular, moreover we clearly see that the Gauss-Seidel histogram is more efficient and faster than the Jacobs histogram.

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Appendix A

```
%Iterative Solutions of linear equation:Jacobi Method
%Linear system:Au=B
%Coefficient matrix A,right-hand side vector B
A=[4 -1 -1 0;-1 4 0 -1;-1 0 4 -1;0 -1 -1 4]
B=[1/3;4/3;0;1/3];
%Set initial value of u to zero column vector
u0=zero(1,4);
%Set Maximum iteration number k_max
K_max=6;
%Set the convergence control parameter erp
erp '=='0.0001;
%Show the q matrix
%loop for iterations
for K=1:K_max
   for i=1:4
      s=0,0;
      for j=1:4
    ifj==i
       continue
     "else"
       s=s+A(i,j)*u0(j);
  end
     end
     u1(i)=(B(i)-s)/A(i,j)
   end
   if norm(u1-u0)<erp
     break
   else
   u0=u1
```

```
    end
"end"
%show the final solution
u=u1
%show the total iteration number
n_iteration=k;
```

Appendix B

```
clear,clc
format compact
%% Read or Input any square Matrix
A=[4 -1 -1 0;
     -1 4 0 -1;
    -1 0 4 -1;
    0 -1 -1 4];%constants vector
C=[1/3;4/3;0;1/3];%constant vector
n= length(C);
X= zeros(n,1);
Error_eval=ones(n,1);
%%Check if the matrix A is diagonally dominant
for i=1:n
    j=1:n;
    j(i)=[];
    B=abs(A(i,j));
    Check(i)=abs(A(i,i))-sum(B);%Is the diagonal value greater than the
  remaining row values combined?
     ifCheck(i)<0
      fprintf('The matrix is not strictly diagonally dominant at row
    %2i\n\n',i)
end
end
%%Start the Iterative method
iteration=0;
while max(Error_eval)>0.001
    iteration=iteration+1;
    Z=X;%save current values to calculate error later
    for i=1:n
        j=1:n;%define an array of the coefficients'elements
        j(i)=[]; %eliminate the unknow's coefficient from the remaining
    coefficients
          Xtemp=X; %copy the unknows to a new variable
          Xtemp(i)=[]; %eliminate the unknows under question from the set
    of values
         X(i)=(C(i)-sum(A(i,j)*Xtemp))/A(i,i);
    end
```

```
    Xsolution(:;iteration)=X;
    Error_eval=sqrt((X-Z).^2);
end
%%Display Results
GaussSeidelTable=[1:iteration;Xsolution]'
MaTrIx=[A X C];
```